

Tractable measure of nonclassical correlation: Use of truncations of a density matrix

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Abstract

For the Oppenheim-Horodecki paradigm of nonclassical correlation, we propose a measure using truncations of a density matrix down to individual eigenspaces. It is computable within polynomial time in the dimension of the Hilbert space.

Keywords: nonclassical correlation, computational tractability

1. Introduction

Classical/nonclassical separation of correlations between subsystems of a bipartite quantum system has been an essential and insightful subject in the quantum information theory. The entanglement paradigm [1, 2] is based on the state preparation stage: any quantum state that cannot be prepared by local operations and classical communications (LOCC) [3] is entangled. There are paradigms [4, 5, 6, 7] based on post-preparation stages, which use different definitions of classical and nonclassical correlations. On the basis of the Oppenheim-Horodecki definition [6, 8, 9], a quantum bipartite system consisting of subsystems A and B is (properly) classically correlated if and only if it is described by a density matrix having a product eigenbasis (PE), $\rho_{\text{PE}}^{\text{AB}} = \sum_{j,k=1,1}^{d^A, d^B} e_{jk} |v_j^A\rangle\langle v_j^A| \otimes |v_k^B\rangle\langle v_k^B|$, where d^A (d^B) is the dimension of the Hilbert space of A (B), e_{jk} is the eigenvalue of $\rho_{\text{PE}}^{\text{AB}}$ corresponding to an eigenvector $|v_j^A\rangle\otimes|v_k^B\rangle$. Thus, a quantum bipartite system consisting of subsystems A and B is nonclassically correlated if and only if it is described by a density matrix having no product eigenbasis. In this Letter, we follow this classical/nonclassical separation.

This definition was introduced in the discussions by Oppenheim *et al.* [6, 8] on information that can be localized by applying closed LOCC (CLOCC, a branch family of LOCC) operations. The CLOCC protocol allows only local unitary operations and operations to send subsystems through a complete dephasing channel. The clas-

sical/nonclassical separation is linked to a localizable information under the zero-way CLOCC protocol in which coherent terms are deleted completely by local players before communicating under CLOCC. A bipartite state with a product eigenbasis carries information completely localizable under zero-way CLOCC. The nonlocalizable information under zero-way CLOCC is a measure of nonclassical correlation.

Other measures [10, 11, 12, 13] than the nonlocalizable information were later proposed on the basis of the same definition of classical/nonclassical correlations. In particular, Piani *et al.* [13] designed a measure which vanishes if and only if a state has a product eigenbasis. It is in a similar form as quantum discord [5] and defined as a distance of two different quantum mutual informations that is minimized over local maps associated with local positive operator-valued measurements [14]. A problem in a practical point of view is that the original nonlocalizable information and the Piani *et al.*'s measure both require expensive computational tasks to take minimums over all possible local operations. A similar difficulty exists in Groisman *et al.*'s measure [10] for which a minimization is actually required to find a proper Schmidt basis (or dephasing basis) used to compute the measure in degenerate cases (namely, the cases where the eigenbases of one or both of the reduced density matrices of the state are not unique). A variant of Groisman *et al.*'s measure, the measurement-induced disturbance measure [15, 16], faces the same difficulty.

In our previous work [11], an entropic measure G based on a sort of game to find the eigenvalues of a reduced density matrix from the eigenvalues of an original density matrix was proposed. This measure can be computed within

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a finite time although it does not have a perfect detection range. Its computational cost is exponential in the dimension of the Hilbert space. One way [17] to achieve a polynomial cost is to introduce carefully-chosen maps similar to positive-but-not-completely-positive maps [2, 18]. We pursue a different way in this letter.

Here, we introduce a measure of nonclassical correlation for a bipartite state using the eigenvalues of reduced matrices obtained by tracing out a subspace after certain truncations of a density matrix. Its construction is rather simple as we see in Definition 2 of Section 3. The computational cost is shown to be polynomial in the dimension of the Hilbert space. Although the measure is imperfect in the detection range and possesses no additivity property, it is practically useful as an economical measure invariant under local unitary operations.

This letter is organized as follows. We begin with a brief overview of the measure G in Section 2. The measure M is introduced and its properties are investigated in Section 3. An extension the detection range and the operational meaning of M are discussed in Section 4. Section 5 summarizes this work.

2. Brief overview of the measure by partitioning eigenvalues

We first briefly overview the measure G , an existing measure computable in finite time. In the context of bipartite splitting, it is defined as the minimized discrepancy between the set of the mimicked eigenvalues of a local system (say, subsystem A), $\{\tilde{e}_i\}_{i=1}^{d^A}$, and the set of the genuine eigenvalues of the local system, $\{e_i\}_{i=1}^{d^A}$. Here, \tilde{e}_i 's are calculated by (i) partitioning the $d^A \times d^B$ eigenvalues of the original bipartite state ρ^{AB} into d^A sets; and (ii) calculating the sum of the elements for each set. The discrepancy in view from one side (from Alice's side in this context) is defined as

$$F^A(\rho^{AB}) = \min_{\text{partitionings}} \left| \sum_i (\tilde{e}_i \log_2 \tilde{e}_i - e_i \log_2 e_i) \right|.$$

Similarly $F^B(\rho^{AB})$ is defined. The measure is defined as

$$G(\rho^{AB}) = \max[F^A(\rho^{AB}), F^B(\rho^{AB})].$$

A drawback of the measure is that the number of combinations of eigenvalues that should be tried in the minimization is $d^A d^B C_{d^B} \times (d^A - 1) d^B C_{d^B} \times \dots \times (d^A - (d^A - 1)) d^B C_{d^B} = (d^A d^B)! / (d^B!)^{d^A} \simeq 2^{d^A d^B \log_2 d^A}$ when the subsystem of concern is A [$(d^A d^B)! / (d^A!)^{d^B} \simeq 2^{d^A d^B \log_2 d^B}$ when it is B]. Indeed, this complexity is better in practice than that

for minimization over all certain local operations required for calculating quantum deficit [6], quantum discord and Piani *et. al.*'s measure. The complexity of a minimization over all local operations for a subsystem, say A, is $O[\text{poly}(d^A, d^B) \times 2^{(d^A)^2 \log_2 c}]$ with c the number of values tried for each parameter of a local operation. The complexity for computing G is smaller in the range $d^A, d^B \lesssim c$. Computing G is, however, still very expensive.

3. Measure based on partial traces of truncated density matrices

A measure that is computable within realistic time is desired for practical use. We introduce in the following a measure that achieves a realistic computational time, namely, polynomial time in the dimension of the Hilbert space.

3.1. Introduction of the measure

Let us begin with a basic definition.

Definition 1. Let us write the eigenspace corresponding to the eigenvalue η of the bipartite density matrix ρ^{AB} as $\text{span}\{|v_k^\eta\rangle\}_{k=1}^{d^\eta}$ where d^η is the dimension of the eigenspace and $|v_k^\eta\rangle$'s are the eigenvectors. Let us define a “truncated” density matrix down to the η eigenspace as

$$\tilde{\rho}^\eta = \eta \sum_{k=1}^{d^\eta} |v_k^\eta\rangle\langle v_k^\eta|.$$

Proposition 1. Consider $\tilde{\rho}^\eta$ introduced above. The eigenvalues of the reduced matrix $\text{Tr}_B \tilde{\rho}^\eta$ ($\text{Tr}_A \tilde{\rho}^\eta$) of the system A (B) are integer multiples of η if ρ^{AB} has a product eigenbasis.

Proof. Suppose ρ^{AB} has a product eigenbasis $\{|a_i\rangle\}_{i=1}^{d^A} \times \{|b_j\rangle\}_{j=1}^{d^B}$. Then $\tilde{\rho}^\eta$ becomes $\eta \sum_{k=1}^{d^\eta} |a_k^\eta\rangle\langle a_k^\eta| \otimes |b_k^\eta\rangle\langle b_k^\eta|$ where $|a_k^\eta\rangle$'s ($|b_k^\eta\rangle$'s) are some d^η vectors, with possible multiplicities, found in $\{|a_i\rangle\}_{i=1}^{d^A}$ ($\{|b_j\rangle\}_{j=1}^{d^B}$). Since $|a_i\rangle$'s ($|b_j\rangle$'s) are orthogonal to each other, it is now easy to find that the proposition holds. \square

Remark 1. The sum of the eigenvalues of $\text{Tr}_B \tilde{\rho}^\eta$ and that of $\text{Tr}_A \tilde{\rho}^\eta$ are both equal to $\text{Tr} \tilde{\rho}^\eta = \eta d^\eta$. This property is tacitly used in the proofs we provide hereafter.

Now we define the new measure of nonclassical correlation on the basis of Proposition 1.

Definition 2. For a given η , consider the eigenvalues of the reduced matrix $\text{Tr}_B \tilde{\rho}^\eta$. For each of the ($\dim \text{Tr}_B \tilde{\rho}^\eta$) eigenvalues, the discrepancy from the nearest integer multiples of η is calculated. Let us write the sum of the discrepancies as s^η for a given η . The measure of nonclassical correlation from the view of the subsystem A is defined as $M^A(\rho^{AB}) = \sum_\eta s^\eta$. In the same way, $M^B(\rho^{AB})$ is defined. The new measure is defined by their average

$$M(\rho^{AB}) = \frac{1}{2}[M^A(\rho^{AB}) + M^B(\rho^{AB})].$$

The improvement in complexity is significant as we mentioned: this measure is computed by testing eigenvalues of at most $2d^A d^B$ reduced matrices. The total complexity is dominated by the complexity of diagonalizing the original density matrix, $O(d^{A^3} d^{B^3})$, which is larger than the complexity of tracing out a subsystem for each truncated density matrix. In the exceptional cases where $d^A > d^{B^2}$ or $d^B > d^{A^2}$, the complexity of diagonalizing all the reduced matrices, $O[\max(d^{A^4} d^B, d^A d^{B^4})]$, becomes the largest cost. Thus the total complexity, namely, the number of the floating-point operations in total to compute the measure, is

$$O[\max(d^{A^3} d^{B^3}, d^{A^4} d^B, d^A d^{B^4})].$$

Example. Consider the two-qubit state

$$\varsigma = \frac{1}{2}(|00\rangle\langle 00| + |1+\rangle\langle 1+|) \quad (1)$$

with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. This has no product eigenbasis while it is separable. The nonzero eigenvalue of ς is $1/2$ with the multiplicity two. The eigenvalue of $\text{Tr}_B \varsigma$ is $1/2$ with the multiplicity two; this leads to $M^A(\varsigma) = 0$. The eigenvalues of $\text{Tr}_A \varsigma$ are $(2 \pm \sqrt{2})/4$; this leads to $M^B(\varsigma) = [2 \times 1/2 - (2 + \sqrt{2})/4] + [(2 - \sqrt{2})/4 - 0 \times 1/2] = 1 - \sqrt{2}/2$. Therefore $M(\varsigma) = (2 - \sqrt{2})/4$.

Remark 2. For any pure state $|\varphi\rangle^{AB}$, $M(|\varphi\rangle^{AB}\langle\varphi|) = 0$ if and only if $|\varphi\rangle^{AB}$ is a product state. This is clear considering the Schmidt decomposition of $|\varphi\rangle^{AB}$.

3.2. The measure is not perfect in the detection range

We have achieved a reduction in the complexity of detecting a nonclassical correlation by introducing the measure M . The measure does not have a perfect detection range as is expected from the fact that it does not test all the local bases unlike other expensive measures like (the minimized) quantum discord (with both-side test) or

zero-way quantum deficit. For example, the measure M vanishes for the two-qubit state

$$\sigma = \frac{1}{6}(|00\rangle\langle 00| + 2|01\rangle\langle 01| + 3|1+\rangle\langle 1+|). \quad (2)$$

Nevertheless, this state has no product eigenbasis because $|0\rangle\langle 0|$ and $|+\rangle\langle +|$ cannot be diagonalized in the same basis.

Consequently, what one can claim is that a state for which the measure does not vanish is in the outside of the set B of the states for which the measure M vanishes, and hence in the outside of the set C of the states having a product eigenbasis, as illustrated in Fig. 1. Note that the set B includes some inseparable states while C does not. For example, the measure M vanishes for the state τ , which is represented as a density matrix acting on the (3×3) -dimensional Hilbert space:

$$\tau = \frac{1}{3}(|\phi\rangle^{AB}\langle\phi| + |\psi\rangle^{AB}\langle\psi| + |\xi\rangle^{AB}\langle\xi|) \quad (3)$$

with

$$\begin{aligned} |\phi\rangle^{AB} &= \frac{|0\rangle^A|1\rangle^B + |1\rangle^A|0\rangle^B}{\sqrt{2}}, \\ |\psi\rangle^{AB} &= \frac{|1\rangle^A|2\rangle^B + |2\rangle^A|1\rangle^B}{\sqrt{2}}, \\ |\xi\rangle^{AB} &= \frac{|2\rangle^A|0\rangle^B + |0\rangle^A|2\rangle^B}{\sqrt{2}}. \end{aligned}$$

This is because the nonzero eigenvalue of τ is $1/3$ with the multiplicity three and the eigenvalue of $\text{Tr}_B \tau = \text{Tr}_A \tau$ is also $1/3$ with the multiplicity three. The state τ is inseparable because its partial transposition $(I \otimes T)\tau$ (here, T is the transposition map) has the eigenvalues $-1/6, 1/6$ and $1/3$ with multiplicities two, six and one, respectively. Thus it has been found that M cannot detect nonclassical correlation of this negative-partial-transpose (NPT) state. This example however does not weaken the measure M very much as there is a way to extend the detection range simply, which we will discuss later in Section 4.

3.3. Relative detection ability

One can compare the measures M and G in their detection ability using the state σ given in Eq. (2) and another particular state for a couple of qubits.

The measure M is vanishing for σ while G is nonvanishing. We have $G(\sigma) = H(1/3) - H[(6 - \sqrt{10})/12] \approx 0.129$ where $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy function.

On the other hand, M is nonvanishing for the state $\sigma' = |\phi\rangle\langle\phi|/2 + (|01\rangle\langle 01| + |10\rangle\langle 10|)/4$ with $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, while G is vanishing. We have $M^A(\sigma') = M^B(\sigma') = 1/2$ and $M(\sigma') = 1/2$.

Therefore, M is neither stronger nor weaker than G in detecting nonclassical correlations.

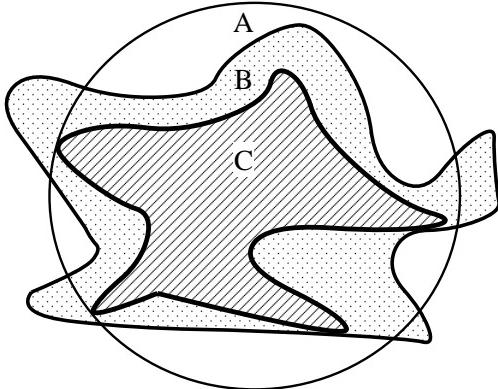


Figure 1: The hierarchy of the quantum states. A: The convex set of the separable states. B: The nonconvex set of the states for which the measure M vanishes. C: the nonconvex set of the states having a product eigenbasis ($C \subset A \cap B$). Internal geometric structures of each set are not depicted, which should wait for future investigations.

3.4. Invariance under local unitary operations

The measure M is invariant under the local unitary operations $\mathcal{U}^A \otimes \mathcal{U}^B : \rho \mapsto (U^A \otimes U^B)\rho(U^A \otimes U^B)^\dagger$.

This is easily verified from the fact that (i) $\text{Tr}_B \tilde{\rho}^\eta$ and $\text{Tr}_B(\mathcal{U}^A \otimes \mathcal{U}^B)(\tilde{\rho}^\eta)$ have the same eigenvalues, and similarly, (ii) $\text{Tr}_A \tilde{\rho}^\eta$ and $\text{Tr}_A(\mathcal{U}^A \otimes \mathcal{U}^B)(\tilde{\rho}^\eta)$ have the same eigenvalues.

3.5. Additivity properties do not hold

The measure M has neither the subadditivity property nor the superadditivity property as we prove below. It is also shown to be not weakly additive.

Here, let us denote a splitting of a system considered for the measure M by a subscript $\cdot|$.

Proposition 2. *Neither the subadditivity*

$$M_{AC|BD}(\rho^{AB} \otimes \sigma^{CD}) \leq M_{A|B}(\rho^{AB}) + M_{C|D}(\sigma^{CD})$$

nor the superadditivity

$$M_{AC|BD}(\rho^{AB} \otimes \sigma^{CD}) \geq M_{A|B}(\rho^{AB}) + M_{C|D}(\sigma^{CD})$$

holds. In addition, the weak additivity

$$M_{A_1 A_2 \cdots | B_1 B_2 \cdots} \left(\bigotimes_{i=1}^m \rho^{A_i B_i} \right) = m M_{A_1 | B_1}(\rho^{A_1 B_1})$$

does not hold.

Proof. (i) First we prove that subadditivity does not hold. Consider the state $\xi = \sigma^{AB} \otimes \sigma^{CD}$ with the state σ defined by Eq. (2). We have already found $M_{A|B}(\sigma^{AB}) = 0$. Now we calculate $M_{AC|BD}(\xi)$. The state ξ has the eigenvalues

$e = 0, 1/36, 1/18, 1/12, 1/9, 1/6$ and $1/4$. Let us write the truncated density matrix down to the e -eigenspace as $\tilde{\xi}^e$. We have $\tilde{\xi}^{1/12} = (|001\rangle\langle 001 + | + |1+00\rangle\langle 1+00|)/12$. This leads to $\text{Tr}_{AC}\tilde{\xi}^{1/12} = (|0+\rangle\langle 0+| + | + | + 0\rangle\langle +0|)/12$ whose eigenvalues are $1/24, 1/8$ and 0 (with the multiplicity two). Similarly, $\tilde{\xi}^{1/6} = (|011\rangle\langle 011 + | + |1+01\rangle\langle 1+01|)/6$ and $\text{Tr}_{AC}\tilde{\xi}^{1/6}$ has the eigenvalues $1/12, 1/4$ and 0 (with the multiplicity two). For other $\tilde{\xi}^e$'s, $\text{Tr}_{AC}\tilde{\xi}^e$ has the eigenvalues equal to e . Therefore, $M_{AC|BD}^{BD}(\xi) = (1+1+2+2)/24 = 1/4$. In addition, it is easy to find $M_{AC|BD}^{AC}(\xi) = 0$ because $\text{Tr}_{BD}\tilde{\xi}^e$ has the eigenvalues equal to integer multiples of e for every e . Consequently, $M_{AC|BD}(\xi) = 1/8$, which is larger than $M_{A|B}(\sigma^{AB}) + M_{C|D}(\sigma^{CD}) = 0$. This is a counterexample to subadditivity.

(ii) Second we prove that superadditivity does not hold. Consider the state $\xi' = \sigma^{AB} \otimes \sigma^{CD}$ with the state σ defined in Eq. (1). We have already found $M_{A|B}(\sigma^{AB}) = (2 - \sqrt{2})/4$. Now we calculate $M_{AC|BD}(\xi')$. The state ξ' has the eigenvalue $1/4$ with the multiplicity four and 0 with the multiplicity 12. We have $\text{Tr}_{AC}\xi' = (|0\rangle\langle 0| + |+\rangle\langle +|)^{\otimes 2}/4$ whose eigenvalues are $(3 \pm 2\sqrt{2})/8$ and $1/8$ with the multiplicity two. This results in $M_{AC|BD}^{BD}(\xi') = [3/4 - (3 + 2\sqrt{2})/8] + (3 - 2\sqrt{2})/8 + 1/8 + 1/8 = 1 - \sqrt{2}/2$. In addition, we have $\text{Tr}_{BD}\xi' = (|0\rangle\langle 0| + |1\rangle\langle 1|)^{\otimes 2}/4$ whose eigenvalues are $1/4$ with the multiplicity four. This results in $M_{AC|BD}^{AC}(\xi') = 0$. Consequently, $M_{AC|BD}(\xi') = (2 - \sqrt{2})/4$, which is less than $M_{A|B}(\sigma^{AB}) + M_{C|D}(\sigma^{CD}) = 1 - \sqrt{2}/2$. This is a counterexample to superadditivity.

(iii) The above counterexamples shown in (i) and (ii) are also counterexamples to weak additivity. \square

4. Discussions

The main aim of introducing the measure M has been the computational tractability. Indeed, it is possible to decide whether a given density matrix has a product eigenbasis within polynomial time simply by diagonalization if there are only nondegenerate eigenvalues for the matrix, but it is, so far, not known to be possible if there are degenerate eigenvalues. Some existing measures, namely, Groisman *et al.*'s measure [10] and its variants, are computable within polynomial time if the dephasing basis is uniquely determined while it is not otherwise. The measure M is, in contrast, always computable within polynomial time.

As is expected for a tractable measure, M is imperfect in its detection range. The measure has been found to vanish not only for the state (2) but for the NPT state (3). It is preferable for a measure of nonclassical correlation not

to vanish for obviously nonclassically correlated states. One may take the average of M and some entanglement measure to produce another measure of nonclassical correlation easily, which is nonvanishing for entangled states detectable by the entanglement measure. This approach was previously introduced in Ref. [17] to combine two different measures of nonclassical correlations.

One can use any entanglement measure for this purpose. As an example, one may use the logarithmic negativity [19] $L(\rho^{AB}) = \log_2[2N(\rho^{AB}) + 1]$ with $N(\rho^{AB})$ the negativity [20, 21] [namely, the absolute value of the sum of the negative eigenvalues of $(I \otimes T)\rho^{AB}$]. The convex combination $M_L = pM + (1 - p)L$ with $0 < p < 1$ is obviously stronger than M and L since they are neither stronger nor weaker to each other. It is, of course, invariant under local unitary operations.

The evaluations of the measure M have been performed on its mathematical properties and computational cost. Let us discuss a rather conceptual problem. It is often of general interest to find an operational (i.e., process-based) meaning of a measure to justify the quantification. In this sense, the following argument may support the measure M .

Suppose a dealer has a source generating quantum systems that are in average described by the density matrix $\rho = \sum_{\eta} \tilde{\rho}^{\eta}$. The source each time generates a system described by a state in an η eigenspace of ρ . The dealer each time sends Alice and Bob the corresponding parts of the generated system with the value of η attached. The dealer may modulate the source by a certain unitary operation common for all the times. The eigenvalues η of a density matrix ρ is unchanged under unitary operations. Therefore each $\tilde{\rho}^{\eta}$ can be regarded as a data set carried in the “channel” η , namely the eigenspace corresponding to η .

Consider which data can be secure data for Alice in the spectrum of $\text{Tr}_B \tilde{\rho}^{\eta}$, against Bob’s guesswork. The label η of the channel is known. Therefore an eigenvalue written as an integer multiple of η is guessed by Bob within at most d^{η} trials. In contrast, a guesswork to find an eigenvalue that is not an integer multiple of η is computationally intractable for Bob: it takes exponential time in the size of the mantissa portion. The discrepancy between the eigenvalue and the nearest integer multiple of η is in other words the smallest shift from the insecure values. Recalling the definition, the measure M^A can be regarded as the sum of the eigenvalue shifts for Alice over the channels η , which are useful to protect her eigenvalues against Bob’s guesswork.

Similarly, M^B can be regarded as the sum of the eigenvalue shifts for Bob over the channels η , which are useful

to protect his eigenvalues against Alice’s guesswork. The measure M has the meaning of the sum of the shifts averaged over the players. In this way, an operational meaning has been brought to M .

The operational meaning introduced here is, however, presumed to be lost for the stronger measure M_L although it is controversial if the process of taking the average of different measures should be regarded as a nonoperational manipulation.

5. Summary

We have proposed an unconventional measure of nonclassical correlation; its mathematical properties and its operational meaning have been investigated. It is invariant under local unitary operations while it is imperfect in the detection range and it has no additivity property. It is usable for a practical evaluation of quantum states because it is calculated within polynomial time in the dimension of a density matrix.

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